Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 2.38

(a)

$$\begin{bmatrix} 1 \times 5 + 2 \times (-6) & 1 \times 0 + 2 \times 7 \\ 3 \times 5 + (-4) \times (-6) & 3 \times 0 + (-4) \times 7 \end{bmatrix} = \begin{bmatrix} -7 & 14 \\ 39 & -28 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} -7 & 14 \\ 39 & -28 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{bmatrix} =$$

$$\begin{bmatrix} -7 \times 1 + 14 \times 2 & -7 \times (-3) + 14 \times 6 & (-7) \times 4 + 14 \times (-5) \\ 39 \times 1 + (-28) \times 2 & 39 \times (-3) + (-28) \times 6 & 39 \times 4 + (-28) \times (-5) \end{bmatrix} =$$

$$\begin{bmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{bmatrix}$$

(b)

$$BC = \begin{bmatrix} 5 \times 1 + 0 \times 2 & 5 \times (-3) + 0 \times 6 & 5 \times 4 + 0 \times (-5) \\ -6 \times 1 + 7 \times 2 & (-6) \times (-3) + 7 \times 6 & -6 \times 4 + 7 \times (-5) \end{bmatrix} =$$

$$\begin{bmatrix} 5 & -15 & 20 \\ 8 & 60 & -59 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 5 & -15 & 20 \\ 8 & 60 & -59 \end{bmatrix} =$$

$$\begin{bmatrix} 1\times 5+2\times 8 & 1\times(-15)+2\times 60 & 1\times 20+2\times(-59) \\ 3\times 5+(-4)\times 8 & 3\times(-15)+(-4)\times 60 & 3\times 20+(-4)\times(-59) \end{bmatrix} =$$

$$\begin{bmatrix} 21 & 105 & -98 \\ -17 & -285 & 296 \end{bmatrix}$$

Problem 2.39

(a)

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1\times 1+2\times 3 & 1\times 2+2\times(-4) \\ 3\times 1+(-4)\times 3 & 3\times 2+(-4)\times(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7\times 1+(-6)\times 3 & 7\times 2+(-6)\times(-4) \\ (-9)\times 1+22\times 3 & (-9)\times 2+22\times(-4) \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$$

(b)

$$AD = \begin{bmatrix} 1\times 3+2\times 4 & 1\times 7+2\times(-8) & 1\times(-1)+2\times 9 \\ 3\times 3+(-4)\times 4 & 3\times 7+(-4)\times(-8) & 3\times(-1)+(-4)\times 9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -9 & 17 \\ -7 & 53 & -39 \end{bmatrix}$$

$$BD = \begin{bmatrix} 5\times 3+0\times 4 & 5\times 7+0\times(-8) & 5\times(-1)+0\times 9 \\ (-6)\times 3+7\times 4 & (-6)\times 7+7\times(-8) & (-6)\times(-1)+7\times 9 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 35 & -5 \\ 10 & -98 & 69 \end{bmatrix}$$

(c)

Not defined. A 2×3 matrix can be multiplied by a 2×3 matrix.

Problem 2.50

(a)

$$A^2 = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + (-5) \times 3 & 2 \times (-5) + (-5) \times 1 \\ 3 \times 2 + 1 \times 3 & 3 \times (-5) + 1 \times 1 \end{bmatrix} =$$
$$\begin{bmatrix} 11 & -15 \\ 9 & -14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 11 & -15 \\ 9 & -14 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 \times 2 + (-15) \times 3 & 11 \times (-5) + (-15) \times 1 \\ 9 \times 2 + (-14) \times 3 & 9 \times (-5) + (-14) \times 1 \end{bmatrix}$$
$$\begin{bmatrix} -67 & 40 \\ -24 & -59 \end{bmatrix}$$

(b)

$$f(A) = \begin{bmatrix} -67 & 40 \\ -24 & -59 \end{bmatrix} \text{-2} \begin{bmatrix} 11 & -15 \\ 9 & -14 \end{bmatrix} \text{-5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -50 & 70 \\ -42 & -36 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 11 & -15 \\ 9 & -14 \end{bmatrix} \text{-3} \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} +17 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 2.53

For finding the inverse of $A = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$ we assume that:

$$\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, we can solve following linear equations to find inverse of A:

$$\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Converting the left half of the following matrix to row canonical form results in finding the inverse of A on the right half:

$$\left[\begin{array}{cc|cc} 7 & 4 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \rightarrow (\text{-}\tfrac{5}{7} \text{ of } R_1 \text{ added to } R_2) \left[\begin{array}{cc|cc} 7 & 4 & 1 & 0 \\ 0 & \frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{4}{7} & \frac{1}{7} & 0 \\ 0 & 1 & -5 & 7 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -5 & 7 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$$

Inverse of B

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 2 & 0 & -5 & 3 \\ 0 & -1 & -2 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{-5}{2} & \frac{3}{2} \\ 0 & 1 & 2 & -1 \end{array}\right] \rightarrow B^{-1} = \begin{bmatrix} \frac{-5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Inverse of C

$$\left[\begin{array}{cc|cc} 4 & -6 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 4 & -6 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right]$$ $\rightarrow$ A row of zeros means the linear system doesn't have a unique solution, or, equivalently, the matrix C is not invertible or singular.

Problem 2.54

Inverse of A

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 1 & 3 & 7 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 2 & -5 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -5 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Inverse of B

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 4 & -3 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & -1 & -4 & 1 \end{array}\right] \rightarrow$$

Problem 3.67

Inverse of A

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 3 & -4 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & 7 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 12 & -5 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} -8 & 12 & -5 \\ -5 & 7 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Inverse of B

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 10 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & -2 & 1 & 0 \\ 0 & 4 & -10 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & -2 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & -2 & 1 \end{array}\right] \rightarrow$$

B is not invertible.

Inverse of C

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 8 & -3 & 0 & 1 & 0 \\ 1 & 7 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 4 & 3 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 7 & -4 & 2 \\ 0 & 2 & 0 & -5 & 3 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14.5 & -8.5 & 3.5 \\ 0 & 2 & 0 & -5 & 3 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14.5 & -8.5 & 3.5 \\ 0 & 1 & 0 & -2.5 & 1.5 & -0.5 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] C^{-1} = \left[\begin{array}{ccc} 14.5 & -8.5 & 3.5 \\ -2.5 & 1.5 & -0.5 \\ 3 & -2 & 1 \end{array}\right] \rightarrow$$

Problem 3.62

(A)

$$\left[\begin{array}{cccccc} 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 & 5 & 7 \\ 3 & 6 & 4 & 9 & 10 & 11 \\ 1 & 2 & 4 & 3 & 6 & 9 \end{array}\right] \rightarrow \left[\begin{array}{cccccc} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 3 & 1 & 5 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccccc} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & -2 & -4 & -2 \end{array}\right] \rightarrow$$

$$(echelon form:)\begin{bmatrix} \mathbf{1} & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & \mathbf{1} & 1 & 3 & 3 \\ 0 & 0 & 0 & \mathbf{2} & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \text{(row canonical form:)} \begin{bmatrix} 1 & 2 & 0 & 0 & -4 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(B)

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 12 \\ 0 & 0 & 4 & 6 \\ 0 & 2 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix} \to$$

$$\text{(echelon form:)} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\text{(row canonical:)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(C)

$$\begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 8 & 5 & 10 \\ 1 & 7 & 7 & 11 \\ 3 & 11 & 7 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 4 & 6 & 8 \\ 0 & 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \text{(echelon form:)} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\text{(row canonical form:)} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3.65

(a)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

e_1^{-1}: "Interchange R_3 by R_2" e_2^{-1}: "Replace R_2 by $R_2/3$"
e_3^{-1}: "Replace R_2 by $R_1 - 2R_3$"

$$E_1' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_2' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3' = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1E_1' = I, E_2E_2' = I, E_3E_3' = I$$

(c)

Assume the question means apply the operations given to columns instead of rows. f_1: "Interchange C_2 and C_3"

f_2: "Replace C_2 by $3C_2$"

f_2: "eplace C_1 by $C_1 + 3C_2$"

(d)

$$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$F_1 = E_1^T = E_1, F_2 = E_2^T = E_2, F_3 = E_3^T$$

Problem 3.66

Suppose by elementary row operations, we can get $E_q ... E_2 E_1 A = I$, then $A = E_1^{-1} E_2^{-1} ... E_q^{-1}$

(A)

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \rightarrow E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & -2 & 0 & 1 \end{array}\right] \rightarrow E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -0.5 \end{array}\right] \rightarrow E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

(B)

$\left[\begin{array}{cc|cc} 3 & -6 & 1 & 0 \\ -2 & 4 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 3 & -6 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \end{array}\right] \rightarrow$ Row of zeros indicates that the matrix is not invertible, therefore, it can not be written in terms of elementary matrices.

(C)

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ -3 & -7 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 0 & 2 & \frac{3}{2} & 1 \end{array}\right] \rightarrow E_1 = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \rightarrow E_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightarrow E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$C = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(D)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 3 & 8 & 7 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 7 & -3 & 0 & 1 \end{array}\right] \rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow E_4 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$